



THE PRESSURE OF A PUNCH WITH A ROUNDED EDGE ON AN ELASTIC HALF-SPACE†

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The problem of the unilateral contact without friction for a punch, the face of which is characterized by a rapid change in the neighbourhood of the a priori unknown boundary of the contact area, is investigated. Asymptotic formulae are obtained for the function which describes the variation of the contact area and the contact-pressure density in the boundary-layer region. The problem of the behaviour of the contact pressures in the neighbourhood of a smoothed stress concentrator is considered. © 2002 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

We will assume that the punch initially touches the surface of an elastic half-space $x_3 > 0$ over an area ω_0 , bounded by a simple smooth closed contour Γ_0 . Here the face of the punch surface is given by the equation

$$x_3 = -\Phi_\epsilon(x_1, x_2); \quad \Phi_\epsilon(x_1, x_2) = \begin{cases} 0, & (x_1, x_2) \in \omega_0 \\ (2R_\epsilon(s))^{-1}n^2, & n \leq 0 \end{cases} \quad (1.1)$$

where s is the length of the arc, n is the distance (taking the sign into account) measured along the inward normal (with respect to the area ω_0) to the curve Γ_0 , $R_\epsilon(s)$ is the radius of curvature of the side part of the vertical section of the punch and $0 < \epsilon$ is a small parameter. We will henceforth assume that

$$R_\epsilon(s) = \epsilon R_1(s), \quad s \in \Gamma_0 \quad (1.2)$$

The continuous function $R_1(s)$ is assumed to be specified and is independent of the parameter ϵ , so that $R_\epsilon(s) \rightarrow 0$ as $\epsilon \rightarrow 0$. In other words, a punch having a base in the form of a smooth surface when $\epsilon \neq 0$ in the limit becomes a punch with a sharp edge.

We will consider the contact problem of the indentation of the punch described into an elastic semi-infinite body (with Young's modulus E and Poisson's ratio ν) to a depth δ_0 . Assuming that there is no friction between the contacting surfaces this problem can be reduced [1] by means of a Papkovitch–Neuber representation to finding a harmonic function u^ϵ , which vanishes at infinity and which satisfies the following boundary conditions (see, for example, [2, 3])

$$\begin{aligned} u^\epsilon(\mathbf{x}) &\geq \delta_0 - \Phi_\epsilon(x_1, x_2), \quad \partial_3 u^\epsilon(\mathbf{x}) \leq 0 \\ [u^\epsilon(\mathbf{x}) - \delta_0 + \Phi_\epsilon(x_1, x_2)] \partial_3 u^\epsilon(\mathbf{x}) &= 0, \quad x_3 = 0 \quad (x_1, x_2) \in \omega_\epsilon^* \\ \partial_3 u^\epsilon(\mathbf{x}) &= 0, \quad x_3 = 0 \quad (x_1, x_2) \notin \omega_\epsilon^* \end{aligned} \quad (1.3)$$

We denote by ω_ϵ^* the region where the inequality $\delta_0 - \Phi_\epsilon(x_1, x_2) > 0$ is satisfied. The face of the indented punch, outside the closure of the region ω_ϵ^* is situated above the unperturbed boundary of the elastic half-space, and hence contact between the punch and the elastic base is only possible inside the region ω_ϵ^* .

The contact area ω_ϵ itself with an a priori unknown boundary Γ_ϵ is found from the condition for the contact pressures to be positive

$$p^\epsilon(x_1, x_2) = -\alpha^{-1} \partial_3 u^\epsilon(x_1, x_2, 0); \quad \alpha \equiv 2(1 - \nu^2)E^{-1} \quad (1.4)$$

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The contact pressure vanishes on the contour Γ_ε and increases rapidly with distance from Γ_ε , reaching its maximum at a certain small distance (which depends on ε) from the contour Γ_ε .

The contact pressure concentration effect in the axisymmetrical problem for a punch with a face $\Phi_\varepsilon(x_1, x_2) = A(x_1^2 + x_2^2)^n$ as the exponent n increases was first pointed out by Shtayerman [4]. The axisymmetrical contact problem for a cylindrical punch with a rounded edge, in the formulation considered, was solved in [5, 6]. The problem of the contact pressure concentration in the plane and axisymmetrical problems in the neighbourhood of the boundary of the contact area was investigated in [7]. Solutions of the integral equations of contact problems having a boundary form, were constructed in [8, 9]. An analytical solution of the non-axisymmetrical contact problem for a punch with a rounding of special form, elliptical in plan, was obtained in [10]. Numerical calculations of the contact pressure concentration have been carried out (see [11, 12], etc.). A method which combines the numerical and asymptotic approaches was proposed in [13] for the case when the contact area is close to rectangular.

Below, to construct an asymptotic solution of the problem, we will use the asymptotic method developed by Nazarov [14, 15] and previously used to investigate a number of structurally non-linear contact problems [16, 17]. The case considered differs from those mentioned in the fact that the shape of the punch is characterized by a rapid change in the neighbourhood of the required contact boundary.

2. ASYMPTOTIC EXPANSIONS IN THE NEIGHBOURHOOD OF THE BOUNDARY OF THE CONTACT AREA

We will denote by v^0 the solution of the unperturbed problem of the progressive indentation of a punch with a plane base in the form of the region ω_0 into an elastic half-space to a depth δ_0 . In the neighbourhood of the contour Γ_0 we change to coordinates (n, x_3, s) , after which, in planes orthogonal to Γ_0 , we introduce the polar coordinates r and $\varphi \in [0, \pi]$ such that $n = r \cos \varphi$, $x_3 = r \sin \varphi$ and the plane part of the base of the punch is locally specified by the equation $\varphi = 0$.

We know (see, for example, [14, 18, 19]), that for the function v^0 in the region of the contour Γ_0 the following asymptotic formula holds (here and henceforth we use the normalization in [14])

$$v^0(\mathbf{x}) = \delta_0 + \alpha(2\pi^{-1})^{1/2} K_0(s)r^{1/2} \sin(\varphi/2) + Q(r^{3/2}), \quad r \rightarrow 0 \quad (2.1)$$

The quantity $K^0(s)$ has the meaning of the compressive stress intensity factor. The following asymptotic representation follows for the contact pressure density corresponding to the potential v^0 from expansion (2.1)

$$p^0(x_1, x_2) = -(2\pi)^{-1/2} K_0(s)r^{-1/2} + O(r^{1/2}), \quad r \rightarrow 0 \quad (2.2)$$

At the same time, the contact pressure density corresponding to the required solution u^ε of singularly perturbed problem (1.3) on the contour Γ_ε of the contact area ω_ε has no singularity, i.e. the following relations must be satisfied as $r_\varepsilon \rightarrow 0$

$$p^\varepsilon(x_1, x_2) = -(2\pi)^{-1/2} k_\varepsilon(s)r_\varepsilon^{1/2} + O(r_\varepsilon) \quad (2.3)$$

$$u^\varepsilon(\mathbf{x}) = \delta_0 - (2R_\varepsilon(s))^{-1} h_\varepsilon(s)^2 - R_\varepsilon(s)^{-1} h_\varepsilon(s)r_\varepsilon \cos \varphi_\varepsilon + \alpha(2\pi^{-1})^{1/2} 3^{-1} k_\varepsilon(s)r_\varepsilon^{3/2} \sin(3\varphi_\varepsilon/2) + O(r_\varepsilon^2) \quad (2.4)$$

Here r_ε , $\varphi_\varepsilon \in [0, \pi]$ are the polar coordinates in planes orthogonal to the contour Γ_ε .

It is assumed that the contour Γ_ε is described by the equation

$$n = -h_\varepsilon(s), \quad s \in \Gamma_0 \quad (2.5)$$

The function $h_\varepsilon(s)$ is to be determined when constructing the approximate solution of the initial problem.

Note that unlike the factor $K_0(s)$ from expansion (2.2), the quantity $k_\varepsilon(s)$, which occurs in (2.4), is of no great importance for applications. In the problem considered it is important to determine the local maximum of the contact pressures on the edge of the contact area.

3. THE PROBLEM OF UNILATERAL CONTACT FOR THE INNER ASYMPTOTIC REPRESENTATION

We will rewrite the Laplace operator in local coordinates (n, x_3, s) in the neighbourhood of the contour Γ_0 . The Lamé parameters of this curvilinear orthogonal system of coordinates are such that $H_1 = H_2 = 1$, $H_s(n, s) = 1 - \kappa_0(s)n$, where $\kappa_0(s)$ is the curvature of the contour Γ_0 . We will now introduce "extended" coordinates (v is a certain positive parameter)

$$\eta_1 = \epsilon^{-v}n, \quad \eta_2 = \epsilon^{-v}x_3 \tag{3.1}$$

Finally, we expand the differential expression obtained in powers of the parameter ϵ^v , separating the principal term $\epsilon^{-2v}(\partial^2/\partial\eta_1^2 + \partial^2/\partial\eta_2^2)$ – a two-dimensional Laplacian.

Hence, the inner asymptotic representation $w^\epsilon(\boldsymbol{\eta}, s)$ of the potential $u^\epsilon(\mathbf{x})$ must mainly satisfy Laplace's equation in the half-plane $\eta_2 > 0$. The boundary condition when $\eta_2 = 0$ arises from (1.3). From formulae (3.1), (1.1) and (1.2) we have

$$\begin{aligned} w^\epsilon(\eta_1, 0, s) &\geq \delta_0 - \epsilon^{2v-1}\Phi_1(\eta_1, s), \quad \partial_3 w^\epsilon(\eta_1, 0, s) \leq 0 \\ [w^\epsilon(\eta_1, 0, s) - \delta_0 + \epsilon^{2v-1}\Phi_1(\eta_1, s)]\partial_3 w^\epsilon(\eta_1, 0, s) &= 0 \end{aligned} \tag{3.2}$$

Here $\Phi_1(\eta_1, s)$ is a function which defines the form of the base of the punch in the neighbourhood of the contour Γ_0 , where

$$\Phi_1(\eta_1, s) = \begin{cases} 0, & \eta_1 > 0 \\ (2R_1(s))^{-1}\eta_1^2, & \eta_1 \leq 0 \end{cases} \tag{3.3}$$

We obtain the condition imposed on the behaviour of the function $w^\epsilon(\boldsymbol{\eta}, s)$ when $|\boldsymbol{\eta}| = (\eta_1^2 + \eta_2^2)^{1/2} \rightarrow \infty$ as a result of matching it (see [18, 20, 21], etc.) with the function $v^0(\mathbf{x})$, the principal term of the outer asymptotic expansion. Thus, substituting the expression $r = \epsilon^v\rho$ into formula (2.1), from the condition for the leading terms of the asymptotic form to be identical we obtain the asymptotic relation

$$w^\epsilon(\boldsymbol{\eta}, s) = \delta_0 + \epsilon^{v/2}\alpha(2\pi^{-1})^{1/2}K_0(s)\rho^{1/2}\sin(\varphi/2) + O(\rho^{-1/2}), \quad \rho \rightarrow \infty \tag{3.4}$$

Here ρ and φ are polar coordinates associated with the extended coordinates (3.1).

Finally, we determine the value of the parameter v from the condition for the orders ϵ^{2v-1} and $\epsilon^{v/2}$ to be identical in relations (3.2) and (3.4). We have $v = 2/3$.

Relations (3.2) and (3.4) comprise the problem for determining the harmonic function $w^\epsilon(\boldsymbol{\eta}, s)$ in the half-plane $\eta_2 > 0$, which depends parametrically on the coordinate s .

4. VARIATION OF THE BOUNDARY OF THE CONTACT AREA

By Eqs (2.5) and (3.1) the function $h_\epsilon(s)$, which defines the position of the required boundary of the contact area Γ_ϵ (distinguishing the dependence on the parameter ϵ in explicit form), must be represented in the form

$$h_\epsilon(s) = \epsilon^{2/3}h_1(s), \quad s \in \Gamma_0 \tag{4.1}$$

Together with the polar coordinates ρ and φ in the plane of the Cartesian coordinates η_1 and η_2 we will introduce the polar coordinates ρ_h and $\varphi_h \in [0, \pi]$ with a pole at the point $O_1 = (-h_1(s), 0)$.

In view of expansion (3.4), distinguishing the terms that increase at infinity, we put

$$w^\epsilon(\boldsymbol{\eta}, s) = \delta_0 + \epsilon^{1/3}\left[\alpha(2\pi^{-1})^{1/2}K_0(s)\rho_h^{1/2}\sin(\varphi_h/2) + W^0(\boldsymbol{\eta}, s)\right] \tag{4.2}$$

The boundary condition which the harmonic function $W^0(\boldsymbol{\eta}, s)$, which vanishes at infinity, must satisfy, is obtained from (3.2) after substituting expression (4.2) there.

By assumption, contact occurs along the section $\eta_1 \geq -h_1(s)$, i.e. when $\eta_1 \geq -h_1(s)$ the equality $w^\epsilon(\eta_1, 0, s) = \delta_0 - \epsilon^{1/3}\Phi_1(\eta_1, s)$ is satisfied, where the remaining part of the boundary of the half-plane must be stress-free, i.e. $\partial_2 w^\epsilon(\eta_1, 0, s) = 0$ when $\eta_1 < -h_1(s)$. Since the harmonic function $\rho_h^{1/2}\sin(\varphi_h/2)$ satisfies these boundary conditions exactly, we obtain

$$W^0(\eta_1, 0, s) = -\Phi_1(\eta_1, s), \quad \eta_1 \geq -h_1(s); \quad \partial_2 W^0(\eta_1, 0, s) = 0, \quad \eta_1 < -h_1(s) \tag{4.3}$$

Moreover, in the neighbourhood of the point O_1 as $\rho_h \rightarrow 0$, by formula (2.4) we have the following asymptotic behaviour

$$W^0(\eta, s) = -(2R_1(s))^{-1} h_1(s)^2 - \alpha(2\pi^{-1})^{1/2} K_0(s) \rho_h^{1/2} \sin(\varphi_h / 2) + O(\rho_h) \tag{4.4}$$

In other words, the normal derivative of the function $W^0(\eta, s)$ on the boundary of the half-plane $\eta_2 > 0$ has a specified root singularity at the point $\eta_1 = -h_1(s)$. Note also that it is necessary to verify a posteriori that the second equation in boundary condition (3.2) for the function (4.2) holds.

Finally, the quantity $h_1(s)$ is found from the condition that the contact pressure intensity factor, corresponding to the function $W^0(\eta, s)$, must be equal to $-K_0(s)$.

Using the well-known procedure [22, 23], we substitute the harmonic functions $W^0(\eta, s)$ and $\zeta(\eta) = (2\pi)^{-1/2} \rho_h^{-1/2} \sin(\varphi_h / 2)$ into Green's formula for the region, which is half a ring with centre at the point O and radii δ and R . Bearing in mind the equalities $\partial_2 \zeta(\eta_1, 0) = 0$ when $\eta_1 < -h_1(s)$ and $\zeta(\eta_1, 0) = 0$ when $\eta_1 > -h_1(s)$, and also boundary conditions (4.3), we will have

$$\int_{S_\delta^+ \cup S_R^+} (W^0(\eta, s) \partial_\nu \zeta(\eta) - \zeta(\eta) \partial_\nu W^0(\eta, s)) ds_\nu + \int_{-h_1(s)+\delta}^0 \Phi_1(\eta_1, s) \partial_2 \zeta(\eta_1) d\eta_1 = 0$$

Here S_δ^+ and S_R^+ are semicircles with centre at the point O_1 and radii δ and R , lying in the half-plane $\eta_2 > 0$, ∂_ν is the derivative along the outward normal and ds_ν is the differential of the length of the arc.

To evaluate the curvilinear integral along the arc S_δ^+ we use asymptotic formula (4.4). We will estimate the integral along the arc S_R^+ using the asymptotic relations $W^0(\eta, s) = O(\rho_h^{-1/2})$ and $\zeta(\eta) = O(\rho_h^{-1/2})$ as $\rho_h \rightarrow \infty$. Taking the limit as $\delta \rightarrow 0$ and $R \rightarrow \infty$, after simple calculations we obtain the equation

$$-\frac{\alpha}{2} K_0(s) + \frac{1}{2\sqrt{2\pi}} \lim_{\delta \rightarrow 0} \left(\int_\delta^{h_1(s)} x^{-3/2} \Phi_1(x - h_1(s), s) dx - \frac{2}{\sqrt{\delta}} \Phi_1(-h_1(s), s) \right) = 0$$

Integrating by parts taking into account the quantity $\Phi_1(0, s) = 0$, we obtain

$$-\frac{\alpha}{2} K_0(s) + \frac{1}{2\sqrt{2\pi}} \int_0^{h_1(s)} x^{-1/2} \Phi_1'(x - h_1(s), s) dx = 0 \tag{4.5}$$

where the prime denotes the derivative of the function $\Phi_1(\eta_1, s)$ with respect to the first argument.

Substituting the expression for the integrand, given by (3.3), into (4.5), we obtain

$$h_1(s) = \left(-\frac{3}{4} \sqrt{\frac{\pi}{2}} \alpha K_0(s) R_1(s) \right)^{2/3} \tag{4.6}$$

Formulae (2.5), (4.1) and (4.6) mainly determine the position of the required boundary Γ_ε of the contact area ω_ε . We emphasize that relations (4.5) and (4.6) have been obtained by Nazarov's method [14, 15] without direct construction of the function $W^0(\eta, s)$.

5. THE PRESSURE CONCENTRATION IN THE NEIGHBOURHOOD OF THE BOUNDARY OF THE CONTACT AREA

Changing to extended coordinates (3.1), we take the relation $\partial/\partial x_3 = \varepsilon^{-2/3} \partial/\partial \eta_2$ into account and, on the basis of formula (1.4) for the contact pressure density in the neighbourhood of the boundary Γ_ε , we obtain the asymptotic representation

$$p^\varepsilon(x_1, x_2) = -\alpha^{-1} \varepsilon^{-2/3} \partial_2 w^\varepsilon(\eta_1, 0, s) \tag{5.1}$$

The solution of problem (3.2), (3.4) can be obtained in explicit form. In particular, according to well-known results [24], we have (the integral is understood in the sense of the principal Cauchy value)

$$-\varepsilon^{-1/3}\alpha^{-1}\partial_2 w^\varepsilon(\eta_1, 0, s) = -\frac{K_0(s)}{\sqrt{2\pi}} \frac{1}{\sqrt{\eta_1 + h_1}} + \frac{1}{\pi\alpha\sqrt{\eta_1 + h_1}} \int_{-h_1}^0 \frac{\sqrt{t + h_1} \Phi_1'(t, s)}{t - \eta_1} dt \quad (5.2)$$

Note that, when deriving formula (5.2), we took into account the condition for the inner asymptotic representation (5.1) to be matched with the outer representation (2.2).

It is easy to see that the contact pressure (5.1) vanishes when $\eta_1 = -h_1(s)$, provided equality (4.5) is satisfied.

Now substituting expression (3.3) into integral (5.2), we obtain the following asymptotic representation for the contact pressure in the boundary-layer region.

$$p^\varepsilon(x_1, x_2) = \frac{\varepsilon^{-1/3}h_1(s)}{\pi\alpha R_1(s)} F\left(\frac{\eta_1 + h_1(s)}{h_1(s)}\right) \quad (5.3)$$

$$F(\xi) = 2\sqrt{\xi} + (\xi - 1) \ln \frac{|\sqrt{\xi} - 1|}{\sqrt{\xi} + 1} \quad (5.4)$$

The function $F(\xi)$ has a maximum where $\xi_0 \approx 0.695$, equal to $F(\xi) \approx 2.399$. Note that ξ_0 is defined as the root of the equation $F'(\xi) = 0$, which, after some reduction, can be transformed to the form (compare with the results obtained by Rostovtsev [7])

$$\xi^{-1/2} \operatorname{th} \xi^{-1/2} = 1 \quad (5.5)$$

Here the following equality holds

$$F(\xi_0) = 2\xi_0^{-1/2} \quad (5.6)$$

Formulae (4.5) and (5.3) to a first approximation solve the problem of the contact pressure concentration in the neighbourhood of the sharp rounded edge of the punch.

6. EXAMPLE

Consider the axisymmetric case, when the contour Γ_0 is a circle of radius b and $R_\varepsilon = \varepsilon R_1$. The pressure under a cylindrical punch with a sharp edge has the following distribution (see, for example, [1])

$$p_0(x_1, x_2) = 2(\pi\alpha)^{-1} \delta_0 (b^2 - x_1^2 - x_2^2)^{-1/2}$$

Hence we determine the value of the compressive stress intensity factor $K_0 = -2\alpha^{-1}\pi^{-1/2}b^{-1/2}\delta_0$. Consequently, from (4.1) and (4.6) we obtain

$$h_\varepsilon = \left(\frac{3\delta_0 R_\varepsilon}{2\sqrt{2}\sqrt{b}} \right)^{2/3} \quad (6.1)$$

On the other hand, according to the well-known solution [6], to determine h_ε we have the equations

$$h_\varepsilon = b(\sec \varphi_0 - 1), \quad \delta_0 = R_\varepsilon^{-1} b^2 \sec \varphi_0 (\operatorname{tg} \varphi_0 - \varphi_0)$$

Retaining only terms of lower powers in the expansions, we obtain the approximate equations $h_\varepsilon = 2^{-1}b\varphi_0^2$ and $\delta_0 = 3^{-1}R_\varepsilon^{-1}b^2\varphi_0^3$. Eliminating the auxiliary variable φ_0 , we arrive at formula (6.1). Note that the roundness of the edge of the punch in principle has no effect on the relation

$$P = 4\alpha^{-1}b\delta_0 \quad (6.2)$$

between the displacement of the punch δ_0 and the force P acting on it.

In the axisymmetric problem considered the idea of the pressure concentration factor

$$k(x_1, x_2) = \pi b^2 P^{-1} p^\varepsilon(x_1, x_2)$$

was introduced in [7]. The following estimate was obtained in [7] for the maximum of the concentration factor

$$k_{\max} = \frac{3\lambda}{4\sqrt{2}} (h_\varepsilon / b)^{-1/2} \quad (6.3)$$

Here λ is the root of the equation $\lambda th\lambda = 1$.

From (5.3) and (5.6) we obtain

$$k_{\max} \approx \frac{bh_\varepsilon}{2\delta_0 R_\varepsilon \sqrt{\xi_0}}$$

Eliminating δ_0 from this relation using (6.1), we obtain formula (6.3), if we bear in mind the equation $\lambda = \xi_0^{-1/2}$.

Note that the form of the rounding, for which it is possible to obtain [10] a simple form of the analytical solution in the neighbourhood of the contour Γ_0 is given by the equation $x_3 = -C(s)|n|^{3/2}$, where $C(s)$ is a determined function. This obviously also explains the considerable disagreement (15% in the boundary-layer region) between the calculated data in [10] and [6].

7. DISCUSSION OF THE RESULTS AND GENERALIZATION

Reverting from the extended coordinate (3.1) to the actual coordinates, we obtain, from formula (4.6) the following relation for the variation of the boundary of the contact area

$$h(s) = \left(-\frac{3\sqrt{\pi}(1-\nu^2)}{2\sqrt{2E}} K_0(s)R(s) \right)^{2/3} \quad (7.1)$$

Here and henceforth, the dependence of the quantities $R(s)$ and $h(s)$ on the parameter ε (the introduction of which was due essentially to the asymptotic approach employed) is not indicated.

In the neighbourhood of the boundary of the contact area we have the following asymptotic representation for the pressure under the punch with a rounded sharp edge, according to relation (5.3),

$$p(x_1, x_2) \approx \frac{Eh(s)}{2\pi(1-\nu^2)R(s)} F\left(\frac{n+h(s)}{h(s)}\right) \quad (7.2)$$

Here n is the distance, taking the sign into account, measured from the contour Γ_0 along the inward normal, and the function $F(\xi)$ is found from formula (5.4).

From relations (7.2) and (5.6) the local maximum of the contact pressures in the neighbourhood of a smooth stress concentrator is given by the formula

$$\max p(x_1, x_2) \approx \frac{Eh(s)}{\pi(1-\nu^2)R(s)\sqrt{\xi_0}} \quad (7.3)$$

where ξ_0 is the root of Eq. (5.5).

Using the method developed by Nazarov [15], we can obtain an estimate of the error of the asymptotic solution constructed with respect to the energy metric. Formulae (7.1)–(7.3) become more accurate the smaller the value of $\max\{|\kappa_0(s)|h(s), h(s)\delta_0^{-1}\}$, where $\kappa_0(s)$ is the curvature of the contour Γ_0 . They have been derived for the case of gradual indentation of the punch. However, it is easy to show that formulae (7.1)–(7.3), and also (4.5), (5.1) and (5.2) remain true in the case of the indentation of a slanting punch provided that the function $K_0(s)$ is negative on Γ_0 . formulae (7.2) and (7.3) are extension of the results obtained by Rostovtsev for the case of axial symmetry.

Note that the assumption that the region ω_0 is simply connected is not essential. Thus, in the problem of the pressure on an elastic half-space by a ring punch with a rounded edge it is necessary to consider two boundary layers (which, in a first approximation, do not interact with one another).

Finally, the formulae obtained also turn out to be suitable in the case of the pressure of a punch on the boundary of an elastic layer of finite thickness. One only needs to determine the function $K_0(s)$ as a solution of the corresponding contact problem for an elastic layer.

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